METRIC AND TOPOLOGICAL SPACES: RE-EXAM 2019/20

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Problem 1 (10%)**.** The discrete metric d_0 attains exactly two different real values. Does there exist a metric function $\varrho\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ which attains exactly 2020 different real values ?
Either prove the non-existence or give an explicit example $(\mathcal{X}, \varrho)$.

Problem 2 (15%)**.** Are the metric topologies induced on the plane $\mathbb{R}^2$ by the metrics d_1 and d_∞ equivalent or not ? (If yes, prove it; or give an example: $U \subseteq \mathbb{R}^2 \mid U \in \mathcal{T}_1$ but $U \notin \mathcal{T}_\infty$, or *vice versa*.)

Problem 3 (15%)**.** Are the subsets $[0, 1)\times[0, 1)$ and $[0, 1]\times[0, 1)$ of Euclidean plane $\mathbb{E}^2$ homeomorphic or not ? (Either prove the impossibility of any $\sim$ or describe one $\sim$ explicitly.)

Problem 4 (10+10%)**.** Find out whether in metric spaces $(\mathcal{X}, d_\mathcal{X})$, each "closed disk" $B^{d_\mathcal{X}}_{\bar{q}}(x_0) = \{x \in \mathcal{X} \mid d_\mathcal{X}(x, x_0) \leqslant q\}$ is closed.

• Does each closed disk always coincide with the closure $\overline{B^{d_\mathcal{X}}_q(x_0)}$ of the open disk of radius q centered at $x_0 \in \mathcal{X}$?
In items (*a*), (*b*) either prove "always" or give a counterexample.

Problem 5 (15%)**.** Let $\mathcal{X}$ be a space such that every continuous function $f\colon \mathcal{X} \to \mathbb{E}^1$ has the property: if $u < w < v$ and $f(x) = u$, $f(y) = v$, then there is $z \in \mathcal{X}$ with $f(z) = w$. Is every such space $\mathcal{X}$ connected or not ? (state and prove, e.g., by counterexample)

Problem 6 (15%)**.** Suppose for every $n \in \mathbb{N}$ that V_n is a non-empty closed subset of a sequentially compact space $\mathcal{X}$ and $V_n \supseteq V_{n+1}$. Prove that $\bigcap_{n=1}^{+\infty} V_n \neq \varnothing$.

Problem 7 (10%)**.** Find a solution $x(s)$ of the integral equation,

$$x(s) = \tfrac{1}{2}\int_0^1 x(t)\,dt + \exp(s) - \tfrac{1}{2}(\exp(1) - 1),$$

by consecutive approximations starting from $x_0(s) = 0$.
Please verify by direct substitution that it satisfies the equation!

Date: June 29, 2020 (08:30–12:00). Good luck & take care ! Reserve: `account?-student.rug.nl`.